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A Class of New Theorems on the Number and Arrangement of the Real Branches of Plane Algebraic Curves.

By L. S. Hulburt.

The maximum number of real branches possible to a plane algebraic curve, of order n, and deficiency p, is, as has been shown by Mr. A. Harnack,* $p+1$. He has also proved the existence of such a curve for every deficiency number p. In a recent article on the real branches of algebraic curves,† Mr. David Hilbert of Königsberg has proved, among other important theorems, that the maximum number of *nested branches* possible to a non-singular curve, which has the maximum number of real branches, is $\frac{1}{2}(n-2)$ when n is even, and $\frac{1}{2}(n-3)$ when n is odd. And furthermore he has shown that, for every order number n, there exists a non-singular curve C_n possessing the following remarkable properties:

1). It has the maximum number of real branches, viz. $p+1$ or $\frac{1}{2}(n-1)(n-2)+1$.

2). It has the maximum number of nested branches, $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-3)$, according as n is even or odd.

3). An ellipse can be drawn which shall enclose one or more of the nested branches of C_n and intersect one of the non-nested closed branches in $2n$ real points, whose order of succession shall be the same upon the branch as upon the ellipse.

By *nested branches* is meant a set of closed branches, of which the first is enclosed by each of the others, the second encloses the first and is enclosed by each of the others, and so on.

In proving the existence of such a curve, Hilbert adopts the method employed by Harnack, which consists in deriving from the equation of a known degenerate curve of order n, an equation representing a curve of the same order, which can be shown to possess the required properties.

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*Mathematische Annalen, Bd. 10. "Ueber die Vieltheiligkeit der ebenen algebraischer Curven."

† Mathematische Annalen, Bd. 38. "Ueber die reellen Züge algebraischer Curven."

It is the object of the present article to extend Hilbert's results to curves having certain singularities, and then to point out the limitations of the method.

I.

Following Hilbert's method, and making use of his results, we shall prove the following

Theorem.—*For every order number n, there exists a curve having* (*a*), *n, or fewer, double points;* (*b*), *the maximum number of real branches, $p+1$; and* (*c*), *the maximum number of nested branches, $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-3)$ according as n is even or odd.*

We consider first the case where the number of double points is n.

Let $\phi_n = 0$ be the equation of a non-singular curve of order n, possessing the three properties which Hilbert's theorem asserts that such a curve may have. Let $f_2 = 0$ be the equation of the ellipse which meets a non-nested closed branch, b, of the curve $\phi_n = 0$, in $2n$ points, whose order of succession is the same upon b as upon the ellipse. Of these points we select any $n+2$, as $p_1, p_2, \dots . p_{n+2}$, and connect them by $n+2$ right lines, in such a way that two lines shall pass through each point, and two points shall lie upon each line. Let the product of the equations of these lines be $l_{n+2} = 0$. We form now the equation

$$\psi_{n+2} \equiv \phi_n f_2 \pm \varepsilon l_{n+2} = 0 \tag{A}$$

where ε is a variable parameter.

For every sufficiently small value of ε this equation represents a curve of order $n+2$, each point of which lies very near a point of the degenerate curve $\phi_n f_2 = 0$. Now the curves $\phi_n f_2 = 0$ and $l_{n+2} = 0$ have each a double point at each of the points $p_1, p_2, \dots . p_{n+2}$. Therefore $\psi_{n+2} = 0$ has a double point at each of these points, that is, $n+2$ *double points in all.*

The ellipse $f_2 = 0$ and the closed branch b form, by their intersection, $2n$ regions, each bounded by a single segment of b and a single segment of the ellipse. One of these regions, which we designate by R, contains, in general, some of the nested branches of $\phi_n = 0$. Let s represent that segment of the ellipse which forms a part of the boundary of R. We next choose the sign of ε so that that portion of the curve $\psi_{n+2} = 0$, which lies very near to the segment s, shall lie wholly within R. That this is always possible arises from the fact that the right lines $l_{n+2} = 0$, having already two points each in common with the ellipse, cannot meet s, except at its extremities. Now the form of equation (A) teaches that, if the curve $\psi_{n+2} = 0$ exists within a given region, G, of the plane, then it exists in any other region, H, only when it is possible to pass from G to H along a continuous path, which meets the lines of the figure in an *even* number of points. By virtue of this principle, it follows that, if the curve $\psi_{n+2} = 0$ exists within any one of the $2n$ regions defined above, *and at the same time lies very near that segment of the ellipse which bounds this region*, then it exists within *each* of these regions. But ε has already been so determined that $\psi_{n+2} = 0$ shall exist within R and in the vicinity of s. Therefore it exists within each of the $2n$ regions. Furthermore, if the right lines had no points in common with either the ellipse or b, the curve would have, within each of these regions, a single closed branch, that is, $2n$ such in all. But in the case under consideration, two of these branches are united at each double point, and a little reflection will make clear that every occurrence of a double point reduces the number of these branches by one. Hence *the curve* $\psi_{n+2} = 0$ *has, in the vicinity of the ellipse and* b, $2n - (n+2)$ *real branches.*

Again, in the immediate neighborhood of each of the remaining $\frac{1}{2}(n-1)(n-2)$ branches of $\phi_n = 0$, there exists a branch of $\psi_{n+2} = 0$. Therefore, *the total number of real branches belonging to* $\psi_{n+2} = 0$ *is* $\frac{1}{2}(n-1)(n-2) + 2n - (n+2) = \frac{1}{2}(\overline{n+2}-1)(\overline{n+2}-2) - (\overline{n+2}) + 1$, *and this is the maximum number.*

The branches of $\psi_{n+2} = 0$ lying in the vicinity of the nested branches of $\phi_n = 0$ are themselves nested branches. That the former curve has yet another nested branch becomes evident on reflecting that the portion of $\psi_{n+2} = 0$, lying very near the segment s of the ellipse, is a portion of a branch, which encloses (or is enclosed by) all the nested branches within the region R, and is, therefore, itself a nested branch. Hence, *the curve* $\psi_{n+2} = 0$ *has* $\frac{1}{2}(n-2) + 1$ *or* $\frac{1}{2}(n-3) + 1$, *that is,* $\frac{1}{2}(\overline{n+2}-2)$ *or* $\frac{1}{2}(\overline{n+2}-3)$ *nested branches, and this is the maximum number.*

Writing n' for $\overline{n+2}$ in the preceding formulas, our argument thus far proves, *for every order number* n', *the existence of a curve* $\psi_{n'} = 0$ *having* n' *double points,* $\frac{1}{2}(n'-1)(n'-2) + 1 = p' + 1$ *real branches, the maximum number, and the maximum number of nested branches,* $\frac{1}{2}(n'-2)$ or $\frac{1}{2}(n'-3)$ *according as* n' *is even, or odd.*

We consider next the case where the number of double points is less than n.

Fixing the attention upon the $2n$ points of intersection of the ellipse with b, we select, not $n+2$ of these points as in the preceding case, but $n+2-r$, and connect them by $n+2-r$ right lines as before. The r remaining right lines are drawn to intersect neither the ellipse nor the branch b. *The curve* $\psi_{n+2}=0$ *has now only* $n+2-r$ *double points.* It has, then, in the vicinity of the ellipse and b, $2n-(n+2-r)$ real branches. The remaining real branches are, in number and arrangement, the same as before. *The curve has, therefore,* $\frac{1}{2}(n-1)(n-2)+2n-(n+2-r)=\frac{1}{2}(n'-1)(n'-2)-(n'-r)+1=p'+1$ *real branches.* And, finally, it is evident that *it has the maximum number of nested branches.*

In order that our argument may still be valid when $n+2-r<3$, it is only necessary to draw the two lines, each of which meets the ellipse in but one double point of $\psi_{n+2}=0$, in such a way that they shall have their second points of intersection with the ellipse upon the *same segment* of the latter.

The preceding results may be summed up as follows:

For every order number n, *there exists a curve*, $\psi_n=0$, *having* (*a*) $n-r$ *double points*, $(r=0,1,2,\ldots n)$, (*b*) $\frac{1}{2}(n-1)(n-2)-(n-r)+1=p+1$ *real branches*, (*c*) $\frac{1}{2}(n-2)$ *or* $\frac{1}{2}(n-3)$ *nested branches, according as n is even or odd.*

II.

Both Harnack and Hilbert, in the articles to which the reference has been made, prove that, for every order number n, there exists a non-singular curve having the maximum number of real branches, $p+1$.

Harnack sets out from a known non-singular curve, $\phi_n=0$, having $p+1$ real branches, one of which is met by a right line, $f_1=0$, in n real points, whose order of succession is the same upon the branch as upon the right line.* He draws, in a suitable manner, $n+1$ other right lines, $l_{n+1}=0$, and forms the equation

$$\psi_{n+1}\equiv\phi_n f_1\pm\epsilon l_{n+1}=0\,.$$

By suitable choice of the magnitude and sign of ϵ, this new curve will have $n-1$ more real branches than has $\phi_n=0$, because the latter curve and the right line, $f_1=0$, form, by their intersection, n regions, in each of which exists a single closed branch of $\psi_{n+1}=0$. The latter has, therefore, $\frac{1}{2}(n-1)(n-2)+n=\frac{1}{2}(\overline{n+1}-1)(\overline{n+1}-2)+1$ real branches, which is the maximum number. The lines $l_{n+1}=0$ are drawn in such a way that the curve $\psi_{n+1}=0$ intersects the right line $f_1=0$ in $n+1$ points whose order of succession is the same upon each. Accordingly, the method may be employed to derive, from $\psi_{n+1}=0$ and $f_1=0$, a new curve, $\psi_{n+2}=0$, possessing the same properties. In this manner Harnack proceeds, step by step, from a known curve with $p+1$ real branches to one of any order having the maximum number of real branches.

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* A non-singular cubic possesses all these properties.

As we have already seen, Hilbert's method differs from this merely in the use of a conic, $f_2=0$, as the auxiliary curve, in place of the right line $f_1=0$. The derived curve is of order $n+2$, has $2n-1$ more real branches than had the original curve $\phi_n=0$, and, hence, has the maximum number of real branches.

The availability of these two methods of proof is due to the fact that the number of new branches formed is, in either case. exactly sufficient to maintain at the maximum the number of real branches of the derived curves.

Now it is worthy of note, and this is the point to which particular attention is called, that, if any curve other than the right line or the conic, be chosen as the auxiliary curve, the derived curve will have less than the maximum number of real branches.

For, let $f_k=0$ represent a curve of order k intersecting a single branch of the non-singular curve $\phi_n=0$ in kn points, whose order of succession is the same upon the one curve as upon the other. Then the equation

$$\psi_{n+k}\equiv\phi_n f_k\pm\varepsilon l_{n+k}=0$$

represents a curve of order $n+k$, and under proper choice of ε, and suitable convention concerning the position of the right lines $l_{n+k}=0$, this derived curve will have a single closed branch in each of the kn regions formed by the intersection of $\phi_n=0$ with $f_k=0$. If now we assume that $\phi_n=0$ has $p+1$ real branches, the total number of real branches belonging to $\psi_{n+k}=0$ is $\frac{1}{2}(n-1)(n-2)+kn$. Let us now assume that this is the maximum number of real branches possible to $\psi_{n+k}=0$. This assumption gives, for the determination of k, the equation

$$\tfrac{1}{2}(n+k-1)(n+k-2)+1=\tfrac{1}{2}(n-1)(n-2)+kn.$$

This is a quadratic in k, whose roots are 1 and 2.

Hence *the auxiliary curve, $f_k=0$ must be either a right line or a conic,* and this was to be proved.

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